Rotation of Space Habitats

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Abstract

Free-floating space habitats can rotate to produce artificial gravity. Structural matter supports the habitat against the resulting centrifugal forces and air pressure. Expanding from a previous work on the energy flow, a physical model is developed to compute the structural mass for habitats of various sizes and shapes, taking into account self-weight in horizontal and vertical support (and bridges between vertical cables). Lower limits on the habitat size are given by the acceptable rotation rate (demanding high rotational radius) and relative shielding mass (demanding high volume-to-surface ratio). Upper limits are posed by co-rotation of the energy collection system (for light and electricity), by the acceptable structural mass, and by the distribution of coolant and light. At small sizes, the dumbbell shape is preferred due to its compactness and adjustable rotational radius. Cylinder and (oblate) spheroid are better for very large habitats due to their lower structural mass, with torus and tube in between. Assuming default parameters (1600m^3 of interior volume, 40kW of power, and 100 tons of interior mass per person, shielding of 5 tons per m^2, ratio of tensile stress to density of $10^5\frac{\text{Nm}}{\text{kg}}$), shielding dominates the mass budget below a population of some 10,000. Co-rotation of the energy collection system is feasible for cylinders with populations up to 200,000 and for dumbbells up to a few million. The minimum total mass per person is achieved in a sphere of 6 million inhabitants, where structural and cooling mass are each 10 tons per person and shielding is 20. Both cooling and structural integrity require ever more mass per person for larger habitats (but even a billion seems possible).

1 Introduction

The settlement of space is an endeavor yet to be undertaken by humanity. While often focused on other planets, free-floating space habitats offer far more diversity, proximity, and growth potential [7]. They utilize the abundant solar power to sustain ecosystems as well as the technical energy demand, and can mainly consist of processed asteroid matter. A shielded hull protects against radiation, rotation provides artificial gravity [5], sunlight can be channeled to the interior by mirrors, and circulation of a coolant to external radiators maintains pleasant temperatures. Small habitats might not need the

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latter two if a transparent hull suffices for lighting and cooling. Gravity is lower closer to the rotational axis, but that offers far more diverse settings than the single value on Earth.

Early design studies of such habitats were done by O'Neill and others in the 1970s [9, 6]. High launch costs have since prevented their realization, but it has also been proposed to start with much smaller and lighter habitats in low Earth orbit [2]. With ongoing launch cost reductions and increasing help from artificial intelligence, the first small habitats in Earth orbit could be built soon [11, 8].

This article investigates the structural mass to support against artificial gravity for various habitat shapes, and expands a previous work on the energy flow [10]. The python implementation is publicly available on github (https://github.com/RainerRolffs/SpaceHabitats).

Sec. 2 describes the structural model, which is used to compare different designs in Sec. 3. The equations are derived in the appendix (Secs. A for geometry, B for gravity, and C for the structural mass).

2 Model Description

The model expands the computations of [10]. Table 1 summarizes the additional input parameters and their default values.

2.1 Habitat

The size of the model is scaled under the assumption of constant usage of the precious shielded interior volume (by division into floors), thus constant population and power (light and electricity) per volume. Population is chosen as the size input parameter due to its intuitive meaning (it can be below 1 for non-human occupation). The volume is given by volume per person, the power consumption by power per person, and the interior mass by interior mass per person. These parameters replace habitat power, power per volume, and interior mass per power from [10]. The equivalent defaults are a population range of 0.025 - 2.5×10^{13} (500 sizes logarithmically spaced), with 1600m^3, 40kW and 100t per person.

2.2 Geometry

Fig. 1 shows a sketch of the shapes that are considered in this work. Energy is collected in the plane facing the Sun, heat is emitted perpendicular to it, facing cold space. The rotation axis is kept along the direction to the Sun, so these orientations do not change. The geometric shapes are parameterized by the ratios of a characteristic length to the rotational radius, the largest axial distance inside the hull. The following shape types are considered:

- **Cylinder:** The cylinder rotates around its central axis. Its length defaults to 1.3 times the radius to maintain rotational stability.

Table 1: Input parameters in addition to [10]

Description	Appendix Symbol	Program Variable	Unit	Default Value
Habitat				
population of the habitat	p	population	(persons)	$2.5 \times (10^{-2} - 10^{13})$
interior volume per person	ν_V	volumePerPerson	m^3	1600
habitat power per person	ν_P	powerPerPerson	W	4×10^4
interior mass per person	ν_I	interiorMassPerPerson	kg	10^5
Geometry				
shape type		shapeType	*Cylinder*, *Tube*, *Oblate*, *Torus*, *Dumbbell*, *Dumbbell-Tube*	*Cylinder*
ratio of cylinder length to rotational radius	ξ_cyl	cylinderLengthToRotRadius		1.3
ratio of tube radius to rotational radius	ξ_T	tubeRadiusToRotRadius		0.1
ratio of oblate minor radius to rotational radius	ξ_obl	oblateMinorToRotRadius		1
ratio of torus habitat radius to rotational radius	ξ_torus	torusHabToRotRadius		0.25
ratio of dumbbell minor radius to rotational radius	ξ_db	dumbbellMinorToRotRadius		0.1
ratio of major to minor dumbbell habitat radii	ν	dumbbellMajorToMinorRadius		1
Gravity Distribution				
constant part of floor height	h_const	constantFloorHeight	m	5
minimum of variable part of floor height	$h_\mathrm{var,min}$	variableFloorHeight	m	5
Structural Integrity				
tensile stress per density	Σ	stressPerDensity	$\frac{\mathrm{Nm}}{\mathrm{kg}}$	10^5
air pressure (already included in the original model for heat absorption by air flow)	p_A	airPressure	bar	0.4
maximum gravity	g_max	maxGravity	$\frac{\mathrm{m}}{\mathrm{s}^2}$	9.81
average distance between vertical cables	d_V	distanceBetweenVerticalCables	m	10
thickness of bridges	b	bridgeThickness	m	1
Energy Collection				
maximum ratio of co-rotating light and electricity radius to co-rotational radius	$\gamma_\mathrm{L,E}$	maxCollectionToCoRotRadius		1
Heat Emission				
maximum ratio of radiator to co-rotational radius	γ_em	maxRadiatorToCoRotRadius		1

- **Tube:** The tube is a cylinder that rotates perpendicular to its length. The endcaps are rounded for a common maximum gravity. The default tube radius to rotational radius is 0.1.
- **Oblate:** Diminishing a sphere's radius along the rotational axis results in an oblate spheroid. The ratio of minor to major radius can be specified (default is 1, a sphere).
- **Torus:** The torus is a ring whose thickness is given by the habitat radius (default is a quarter of the rotational radius).
- **Dumbbell:** Two spheres rotate around their center of mass, connected by cables. An asymmetric dumbbell has spheres of different radii, whose ratio can be specified (default is 1, symmetric). The (minor) habitat radius defaults to 0.1 times the rotational radius.
- **Dumbbell with Tube:** In this case, the spheres are connected by a shielded and pressurized tube (of given radius ratio), instead of merely elevator and cables.

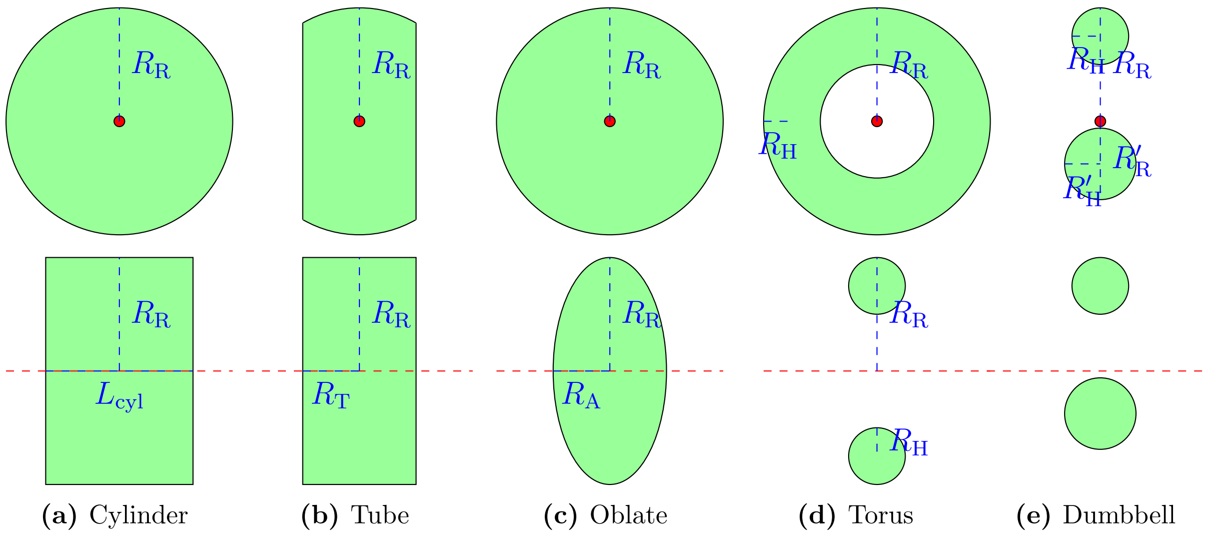

Figure 1: Sketches of the habitat shapes. The upper panels show the view along the axis (red), the lower panels perpendicular to it (tube and dumbbell are seen from the side, while the other shapes are rotationally symmetric). With the axis permanently pointing to the Sun, mirrors and PV panels would extend in the upper panels, the radiator in the lower panels (not shown). The aspect ratios (fractions of rotational radius R_R) are (a) $\xi_{cyl} = 1.3$, (b) $\xi_T = \frac{1}{2}$, (c) $\xi_{obl} = \frac{1}{2}$, (d) $\xi_{torus} = \frac{1}{4}$, and (e) $\xi_{db} = \frac{1}{4}$ and sphere radii ratio of $\sqrt[3]{2}$ (the second sphere has twice the volume).

2.3 Gravity Distribution

The interior volume is divided into floors (either separated or a continuous spiral, connected by elevators and stairs). The floor heights may vary with gravity - this is modeled as a constant component plus a height that is inversely proportional to gravity. Both constant and minimum variable part default to 5m, so the outermost floor is 10m high, while at half the rotational radius the height is 15m.

2.4 Structural Integrity

Structural material is characterized by the ratio of tensile stress to density Σ, where stress is the force that acts on it per cross section. It should be considerably lower than the tensile strength where the material breaks. For example, a tensile stress of a few times $10^8 \frac{\text{N}}{\text{m}^2}$ applied to a mix of materials whose average density is a few times $10^3 \frac{\text{kg}}{\text{m}^3}$ results in the default value of $10^5 \frac{\text{J}}{\text{kg}}$. Steel can hardly reach this default, but lighter temporary materials can surpass it by an order of magnitude (and carbon nanostructures by more than two orders).

In addition to artificial gravity, pressure containment also needs structural mass (proportional to the air mass). The input parameter for air pressure is already included in the original model (where it is used for heat absorption by air flow).

The maximum gravity g_{max} defaults to Earth normal gravity.

Support against gravity is provided either as radial cables (vertically supporting hanging masses) or integrated in hull and ground (horizontally supporting standing masses).

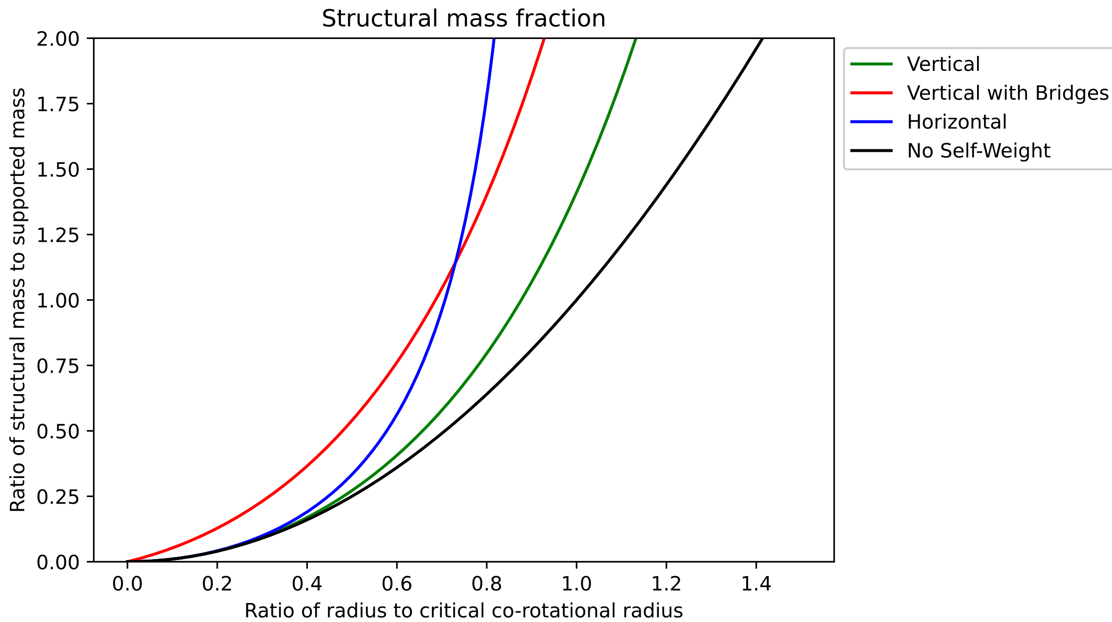

Figure 2: Structural mass per supported mass as function of radius. Horizontal support has more self-weight than vertical support, limiting it to radii below the critical co-rotational radius R_{CoRot}, which grows with the square root of the rotational radius R_{R}. For default parameters and a rotational radius of 100m, R_{CoRot} is 1km. It meets the rotational radius at 10km. For clarity, bridges are exaggerated by setting the distance between the vertical cables to 50m (the default of 10m is close to the pure vertical).

Vertical support has less self-weight (Sec. C), but bridge-like horizontal structures are needed between the vertical cables. The distance they span defaults to 10m, the bridge thickness (height of the horizontal structure or the soil) to 1m. The radiator as well as hull and interior of dumbbell and tube can only be held vertically; else the method with the lower mass is chosen for each floor.

2.5 Energy Collection / Heat Dissipation

Mirrors and PV are decoupled from rotation at radii above $\gamma_{\text{L,E}}$ times the critical co-rotational radius (Sec. C.6, default 1). Non-rotating mirrors must be cylindrically symmetric, PV connected by sliding contacts.

In the model, the whole radiator co-rotates with the habitat. Its extension from the axis is hence limited to γ_{em} times the critical co-rotational radius (default is 1). The length is increased (and the quadratic radiator shape abandoned) when this limit is reached.

3 Results

Containing the air can require more mass than the air itself - irrespective of the air pressure and inversely proportional to the stress per density (the default results in 1.6

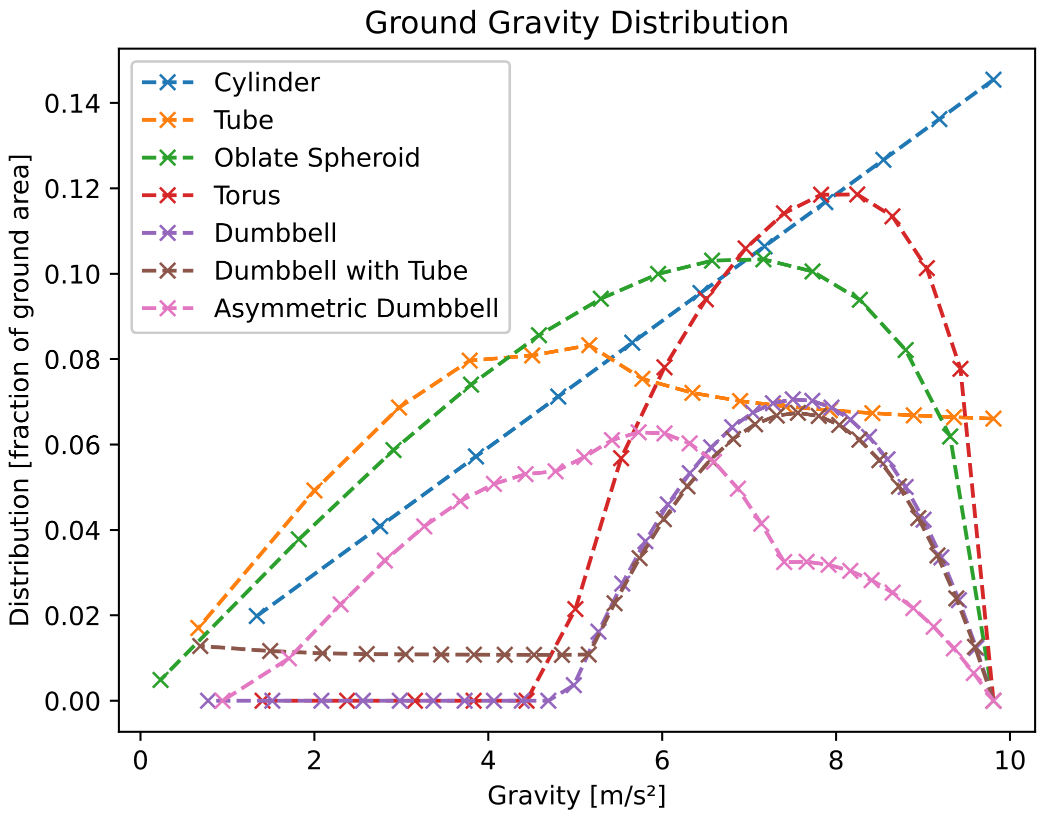

Figure 3: Gravity distribution inside different habitat shapes (examples of medium size, Tab. 3). Each cross represents a floor (connecting lines are only for visual help), indicating the gravity at its bottom (ground) and the fraction of total ground area. Floor heights vary with gravity (constant height of 5m plus a variable height of 5m divided by the ratio of gravity to maximum gravity).

times as much).

The amount of structural mass to counter artificial gravity is proportional to the supported mass and the square of their velocity, plus self-weight (Sec. C). The cross section of a cable (in vertical support) falls with distance from the axis, following a Gaussian shape. The resulting structural mass fraction is shown in Fig. 2.

To compute the total structural mass, the gravity distribution is needed (Sec. B). Fig. 3 shows the ground gravity level for each floor in different habitat shapes.

Each component of the habitat has its own distribution of gravity and mass, yielding a specific structural mass. Their dependence on the habitat size can be seen in Fig. 4. For example, as the hull mass grows with the square of the radius, the structural mass to hold it grows with the third power of the radius, i.e. the hull structural mass per volume is constant until self-weight becomes relevant at large sizes.

The amount of structural mass is comparable to the cooling mass (Fig. 5) - both have low mass below about 10 million inhabitants (10^{10}m^3), where they meet the hull mass at an order of magnitude below the interior mass.

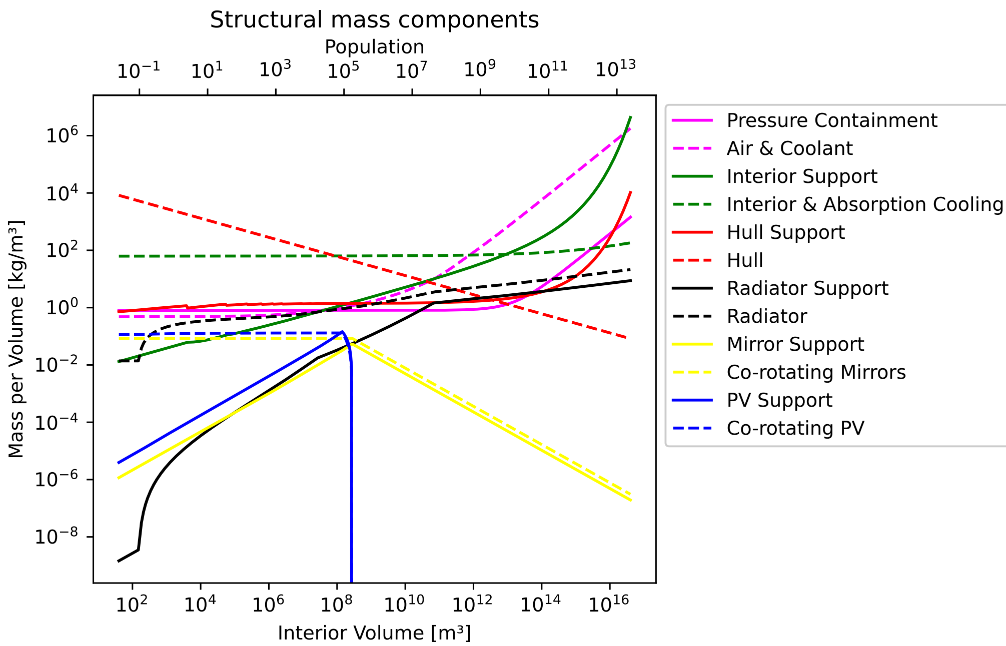

Figure 4: Structural masses with corresponding supported masses per volume in the default case. In energy collection, too large ratios are prevented by abandoning co-rotation, in cooling by limiting the radiator width. Pressure containment does not support against gravity, but against pressure from air and coolant.

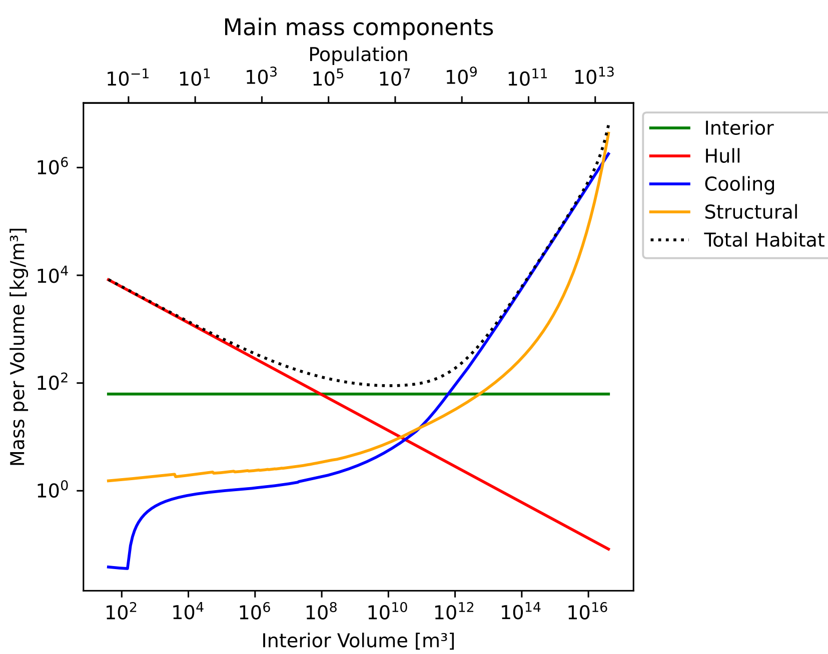

Figure 5: Mass per volume of the main components in the default case. Both cooling and structural mass effectively prevent sizes of more than a billion ($10^{12}\mathrm{m}^3$).

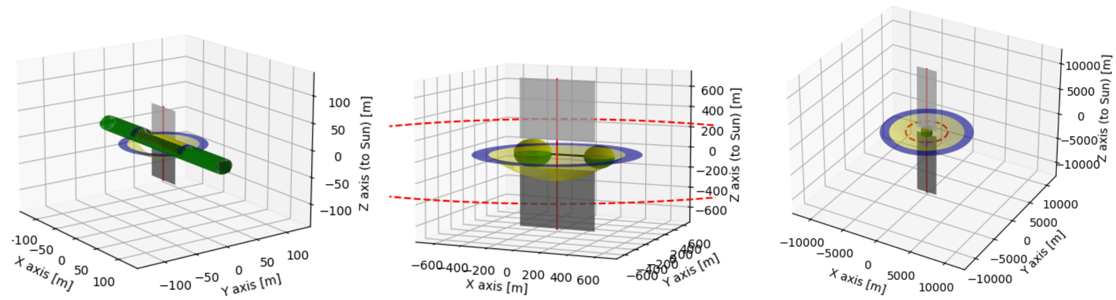

Figure 6: 3-d plots of a small tube, a medium asymmetric dumbbell, and a large cylinder (green: habitat, yellow: mirrors for light collection, blue: PV modules for electricity, gray: radiators, red: rotation axis, dashed red: critical co-rotational radius)

Table 2: Small habitats for around one hundred people (interior volume of $1.6 \times 10^5 m^3$)

Shape	Hull Mass ($\times 10^8$ kg)	Structural Mass ($\times 10^5$kg)	Rot. Radius (m)	# Floors	Avg. Gravity ($\frac{m}{s^2}$)
Cylinder (length to rot.radius 0.5)	1.0	2.9	47	4	3.8*
Tube (tube to rot.radius 0.1)	1.2	8.4	137	10	5.8
Oblate Spheroid (minor to rot.radius 0.25)	1.0	2.7	53	4	3.0*
Torus (hab to rot.radius 0.1)	1.7	14.8	97	2	8.8
Dumbbell (hab to rot.radius 0.2)	0.9	11.4	134	5	7.9
Dumbbell with Tube (tube to rot.radius 0.1)	1.1	8.5	115	8	7.0
Asymmetric Dumbbell (volume ratio 2)	0.9	7.3	117	7	6.2

Notes. * max. gravity reduced by half to keep the rotation rate at about 3rpm

Different shapes of same habitat size are compared in Tables 2, 3, and 4. The size category S (small, Tab. 2) has a population of 100, M (medium, Tab.3) has 10,000 people, and L (large, Tab. 4) has a million (for comparison, interior mass amounts to 10^5kg per person). Aspect ratios are adapted for low enough rotation rate, requiring a minimum radius of around 100m to avoid rotation rates above 3rpm at Earth gravity [3]; thus the maximum gravity of cylinder and oblate spheroid in S is reduced by half. The dumbbell shape has the lowest mass, while providing high gravity and slow rotation. In M and L, the spheroid has the lowest mass (but other shapes might be preferred for gravity distribution). In L, mirrors extend to 4.9km (and PV to 6.1km) from the axis. Only the dumbbell shapes allow full co-rotation of mirrors, while the cylinder has a critical co-rotational radius of 2.7km (demanding stationary mirrors in cylindrical symmetry). This problem is more severe for larger distance from the Sun. Further extending the categories by a factor of 100, XS for one person suffers from relatively high shielding mass, and only a dumbbell (or a sphere with a counter-weight) can have gravity. XL for 100 million seems possible, but is at the limit for structural integrity, cooling and light distribution.

3-d plots of all habitats can be made with the open-source code (examples in Fig. 6).

4 Discussion

Mass can be a proxy for cost, albeit with different weighting for the components. Shielding in the hull can be made of wastes or raw material, while the other components require

Table 3: Medium habitats for around ten thousand people (interior volume of $1.6 \times 10^7 m^3$)

Shape	Hull Mass ($\times 10^9$ kg)	Structural Mass ($\times 10^7$kg)	Rot. Radius (m)	# Floors	Avg. Gravity ($\frac{m}{s^2}$)
Cylinder (length to rot.radius 1.3)	1.8	4.5	158	11	7.2
Tube (tube to rot.radius 0.5)	1.9	4.9	217	15	6.3
Oblate Spheroid (minor to rot.radius 0.5)	1.6	4.0	197	14	6.3
Torus (hab to rot.radius 0.25)	2.5	7.2	259	12	7.6
Dumbbell (hab to rot.radius 0.25)	1.9	10.5	496	22	7.5
Dumbbell with Tube (tube to rot.radius 0.1)	2.3	9.0	462	30	7.0
Asymmetric Dumbbell (volume ratio 2)	1.9	6.5	434	27	5.9

Table 4: Large habitats for around a million people ($1.6 \times 10^9 m^3$)

Shape	Hull Mass ($\times 10^{10}$ kg)	Structural Mass ($\times 10^{10}$kg)	Rot. Radius (m)	# Floors	Avg. Gravity ($\frac{m}{s^2}$)
Cylinder (length to rot.radius 1.3)	3.9	0.8	732	47	7.1
Tube (tube to rot.radius 0.5)	4.0	0.9	1006	64	6.3
Sphere	3.3	0.7	726	46	6.4
Torus (habitat to rot.radius 0.25)	5.3	1.4	1200	52	7.7
Dumbbell (habitat to rot.radius 0.25)	4.2	2.3	2304	99	7.5
Dumbbell with tube (tube to rot.radius 0.1)	4.9	2.0	2144	134	7.1
Asymmetric Dumbbell (volume ratio 2)	4.1	1.3	2012	123	5.8

processing of matter.

Several constraints on the size of a space habitat have been quantified: Lower limits are given by the acceptable hull mass per interior volume (Fig. 5), and by the acceptable rotational rate. Both hull mass and rotational rate depend on shape and aspect ratio, so a compromise must be found between low hull mass (demanding compactness) and low rotational rate (demanding a large rotational radius). With growing size, the volume inside a given rotational radius can be filled starting from points (dumbbell), lines (tube, torus), or a surface (cylinder, oblate spheroid). For the latter, a compact shape (high aspect ratio) is only feasible for large habitats. The rotational rate would be above 3 rpm (rotational radius smaller than 100m) at populations below 2600 (volume $4 \times 10^6 m^3$) for cylinder and sphere, below 10 people ($2 \times 10^4 m^3$) for a tube with 5m radius, and below 30 people ($5 \times 10^4 m^3$) for a torus with 5m radius, while the dumbbell does not have a minimum size.

The hull mass is above the interior mass below a population of 60,000 ($10^8 m^3$). Less shielding might suffice in equatorial low-Earth orbit [4]: a tenth of hull surface density (so half a ton per m^2) shifts this limit to 100 people ($10^5 m^3$), the S size, and allows more elongated shapes. For example, a transparent hull around the small tube could naturally cool and light the interior (dissipating $160 \frac{W}{m^2}$ of hull surface and retrieving light of $400 \frac{W}{m^2}$ of cross section). The ground could wind like a staircase around a central tower, and an additional daily rotation around the tube's central axis would provide a circadian rhythm (requiring sliding contacts to the PV modules).

Upper limits on the habitat size are given by

- **constraints on cooling and lighting:** As derived in [10], cooling mass surpasses the interior mass at around 10^{13}W ($6 \times 10^{11} m^3$, 400 million inhabitants), which happens to be the same size where lighting would take up too much volume for distribution in light channels. A mass reduction in cooling by an order of magnitude is reached by vapor or air instead of liquid, but these cooling methods are limited to below 200 million people ($3 \times 10^{11}\text{m}^3$).

- **by the requirement of co-rotating mirrors:** Mirrors for focusing sunlight onto habitat windows could be non-rotating, but this allows less concentration and requires window rings on cylindrically symmetric shapes such as torus, cylinder, or oblate spheroid. Alternatively, light could be produced electrically. While electric generation can co-rotate below 100,000 inhabitants, the radius of light collection surpasses the critical co-rotational radius at 180,000 inhabitants ($3 \times 10^8\text{m}^3$ and $7 \times 10^9\text{W}$) for default parameters. This limit is stretched to 4 million ($6 \times 10^9\text{m}^3$) for a dumbbell, but shifts to 26,000 ($4 \times 10^7\text{m}^3$) if the cylinder is placed at 2AU from the Sun. 10 times higher tensile strength per density ($\Sigma = 10^6 \frac{\text{m}^2}{\text{s}^2}$) leads to 6 million ($10^{10}\text{m}^3$ and $2 \times 10^{11}\text{W}$). This size is very similar to the spot of minimum mass, where both hull and cooling masses are relatively low.

- **by the acceptable structural mass:** For small habitats, the structural mass to support the hull is similar to the pressure containment - almost proportional to size at 2% of the interior mass for default parameters, as the increasing radius is compensated by the decreasing hull mass per volume (Fig. 4). Support for the interior dominates the structural mass above a population of 100,000 and it surpasses the hull mass at 16 millions. At 3 billions, the rotational radius reaches its co-rotational limit of 10km, and the structural mass reaches the interior mass.

5 Conclusions

To compute the structural mass of rotating space habitats, self-weight in horizontal (standing) and vertical (hanging) support is considered. A critical co-rotational radius is derived, beyond which horizontal support is impossible (and vertical support requires a structural mass of 1.4 times the supported mass).

Different habitat shapes are modeled, from (asymmetric) dumbbell over tube and torus to cylinder and oblate spheroid. Scaled by interior volume, they differ in hull surface area, gravity distribution, and rotational radius (where maximum gravity is reached).

As shielding dominates the mass budget of small habitats, a minimum habitat size depends on the shielding surface density and the acceptable hull mass. A further constraint arises from a maximum rotation rate, so a minimum rotational radius. Small habitats prefer a dumbbell shape due to its flexible rotational radius.

The lowest upper limit on the habitat size is the co-rotation of mirrors, which is needed for focusing on windows. If stationary mirrors focus on a window ring instead, distribution of coolant and light pose the next upper limit, followed by structural integrity of the habitat interior and hull. The requirement of co-rotation also limits the width of the radiator, forcing it to elongate along the rotation axis and thereby worsening the coolant distribution problem.

An optimum size is found in the region between tens of thousands and tens of millions (10^8 - 10^{11}m^3), where the interior mass (the "payload") dominates.

The habitat with the lowest total mass per person is a sphere of 6 million (10^{10}m^3, $2 \times 10^{11}\text{W}$). Its radius is 1.3km, its total mass is $8 \times 10^{11}\text{kg}$, of which almost three quarters are interior mass. For the default dumbbell, the optimum is at half a million ($8 \times 10^8 m^3$, rotational radius of 4.6km) due to the larger structural mass.

Optimizing for cost would shift to slightly lower sizes, since hull shielding can be made from waste and is cheaper.

Appendices

A Habitat Geometry

The interior volume of the habitat is specified by the habitat population p and the volume per person ν_V as $V_\mathrm{H} = p\nu_\mathrm{V}$. The habitat power is $P_\mathrm{H} = p\nu_\mathrm{P}$ (with power per person ν_P), and the interior mass $M_\mathrm{I} = p\nu_\mathrm{I}$ with interior mass per person ν_I.

A.1 Rotational Radius

The rotational radius R_R is the maximum radius from the center of mass within the habitat. Each shape is characterized by an aspect ratio ξ (see Fig. 1): The cylinder length is $L_\mathrm{cyl} = \xi_\mathrm{cyl} R_\mathrm{R}$, the tube radius is $R_\mathrm{T} = \xi_\mathrm{T} R_\mathrm{R}$. The oblate spheroid has a minor radius $R_\mathrm{A} = \xi_\mathrm{obl} R_\mathrm{R}$, and the torus habitat radius is $R_\mathrm{H} = \xi_\mathrm{torus} R_\mathrm{R}$. The dumbbell habitat radius is $R_\mathrm{H} = \xi_\mathrm{db} R_\mathrm{R}$. The two spheres can be of different size, in which case the larger sphere has $R'_\mathrm{H} = \nu R_\mathrm{H}$, defining another aspect ratio ν. Finally, the connection between them can be a tube, combining the two shapes.

The rotational radius is derived from the given volume and the aspect ratios of the geometric shape.

- **Cylinder:** The cylinder habitat volume is $V_\mathrm{H} = \pi R_\mathrm{R}^2 L_\mathrm{cyl} = \pi \xi_\mathrm{cyl} R_\mathrm{R}^3$. It follows that $R_\mathrm{R} = \sqrt[3]{\frac{V_\mathrm{H}}{\pi \xi_\mathrm{cyl}}}$.
- **Tube:** Neglecting the curvature of the endcaps, the length is approximately $2R_\mathrm{R}$ and the volume is approximated as $V_\mathrm{H} = \pi R_\mathrm{T}^2 2R_\mathrm{R} = 2\pi \xi_\mathrm{T}^2 R_\mathrm{R}^3$, so $R_\mathrm{R} = \sqrt[3]{\frac{V_\mathrm{H}}{2\pi \xi_\mathrm{T}^2}}$.
- **Oblate:** With the radius of the minor axis $R_\mathrm{A} = \xi_\mathrm{obl} R_\mathrm{R}$, the volume of an oblate spheroid is $V_\mathrm{H} = \frac{4\pi}{3} R_\mathrm{A} R_\mathrm{R}^2 = \frac{4\pi}{3} \xi_\mathrm{obl} R_\mathrm{R}^3$, i.e. lower than the sphere by the aspect ratio ξ_obl. The rotational radius is $R_\mathrm{R} = \sqrt[3]{\frac{3V_\mathrm{H}}{4\pi \xi_\mathrm{obl}}}$.
- **Torus:** A torus has $V_\mathrm{H} = 2\pi(R_\mathrm{R} - R_\mathrm{H})\pi R_\mathrm{H}^2 = 2\pi^2 \xi_\mathrm{torus}^2 (1 - \xi_\mathrm{torus}) R_\mathrm{R}^3$, so $R_\mathrm{R} = \sqrt[3]{\frac{V_\mathrm{H}}{2\pi^2 \xi_\mathrm{torus}^2 (1-\xi_\mathrm{torus})}}$.
- **Dumbbell:** The habitat volume $V_\mathrm{H} = \frac{4\pi}{3} R_\mathrm{H}^3 + \frac{4\pi}{3} R_\mathrm{H}'^3 = \frac{4\pi}{3}(1 + \nu^3)\xi_\mathrm{db}^3 R_\mathrm{R}^3$ leads to $R_\mathrm{R} = \frac{1}{\xi_\mathrm{db}} \sqrt[3]{\frac{3V_\mathrm{H}}{4\pi(1+\nu^3)}}$.

  In an asymmetric dumbbell, the mass ratio of larger to smaller sphere is $\mu_\mathrm{db} \approx \frac{\sigma_\mathrm{S} A'_\mathrm{S} + \rho_\mathrm{I} V'_\mathrm{H}}{\sigma_\mathrm{S} A_\mathrm{S} + \rho_\mathrm{I} V_\mathrm{H}} = \frac{\sigma_\mathrm{S} R_\mathrm{H}'^2 + \rho_\mathrm{I} \frac{1}{3} R_\mathrm{H}'^3}{\sigma_\mathrm{S} R_\mathrm{H}^2 + \rho_\mathrm{I} \frac{1}{3} R_\mathrm{H}^3} = \frac{\sigma_\mathrm{S} \nu^2 + \rho_\mathrm{I} \frac{1}{3} \nu^3 \xi_\mathrm{db} R_\mathrm{R}}{\sigma_\mathrm{S} + \rho_\mathrm{I} \frac{1}{3} \xi_\mathrm{db} R_\mathrm{R}}$ with the hull surface density σ_S and the interior mass per volume $\rho_\mathrm{I} = \frac{\nu_\mathrm{I}}{\nu_\mathrm{V}}$. The rotational radius of the larger sphere is found by the center of mass $\frac{R'_\mathrm{R} - R'_\mathrm{H}}{R_\mathrm{R} - R_\mathrm{H}} = \mu_\mathrm{db}$, so $R'_\mathrm{R} = R'_\mathrm{H} + \mu_\mathrm{db}(R_\mathrm{R} - R_\mathrm{H}) = (\nu \xi_\mathrm{db} + \mu_\mathrm{db}(1 - \xi_\mathrm{db}))\, R_\mathrm{R}$.
- **Dumbbell with Tube:** A shielded and inhabited connection tube adds volume and mass; the resulting shift of the center of mass is neglected for simplicity, so the

tube length $L_{\mathrm{T}} = R_{\mathrm{R}} - 2R_{\mathrm{H}} + R'_{\mathrm{R}} - 2R'_{\mathrm{H}}$ is evaluated with the above expression for R'_{R}, yielding the ratio of tube length to rotational radius $l_{\mathrm{T}} = \frac{L_{\mathrm{T}}}{R_{\mathrm{R}}} = 1 - 2\xi_{\mathrm{db}} + \nu\xi_{\mathrm{db}} + \mu_{\mathrm{db}}(1 - \xi_{\mathrm{db}}) - 2\nu\xi_{\mathrm{db}} = 1 + \mu_{\mathrm{db}} - \xi_{\mathrm{db}}(2 + \nu + \mu_{\mathrm{db}})$. The model does not account for the case of a negative L_{T}, in which the two spheres partly merge. The total volume is $V_{\mathrm{H}} = \frac{4\pi}{3}R_{\mathrm{H}}^3 + \frac{4\pi}{3}R'^3_{\mathrm{H}} + \pi R_{\mathrm{T}}^2 L_{\mathrm{T}} = \frac{4\pi}{3}(1+\nu^3)\xi_{\mathrm{db}}^3 R_{\mathrm{R}}^3 + \pi\xi_{\mathrm{T}}^2 l_{\mathrm{T}} R_{\mathrm{R}}^3$, thus $R_{\mathrm{R}} = \sqrt[3]{\frac{V_{\mathrm{H}}}{\frac{4\pi}{3}(1+\nu^3)\xi_{\mathrm{db}}^3 + \pi\xi_{\mathrm{T}}^2 l_{\mathrm{T}}}}$.

A.2 Surface

The surface area determines the mass of the hull shielding, $M_{\mathrm{S}} = A_{\mathrm{S}}\sigma_{\mathrm{S}}$ with the surface density σ_{S}. It is already needed for the power of heat conduction through the hull. Seen from the Sun, the habitat spans a certain cross section (which has the same orientation as the cross section of the cooling flow in heat connection).

- **Cylinder:** The cylinder surface is $A_{\mathrm{S}} = 2\pi R_{\mathrm{R}}^2 + 2\pi R_{\mathrm{R}} L_{\mathrm{cyl}} = 2\pi R_{\mathrm{R}}^2(1 + \xi_{\mathrm{cyl}})$. Its cross section is $A_{\perp,\mathrm{H}} = \pi R_{\mathrm{R}}^2$.
- **Tube:** The tube surface is approximated as $A_{\mathrm{S}} = 2\pi(R_{\mathrm{T}}^2 + R_{\mathrm{T}} 2R_{\mathrm{R}}) = 2\pi R_{\mathrm{R}}^2(\xi_{\mathrm{T}}^2 + 2\xi_{\mathrm{T}})$. The cross section is $A_{\perp,\mathrm{H}} = 4R_{\mathrm{T}}R_{\mathrm{R}} = 4\xi_{\mathrm{T}}R_{\mathrm{R}}^2$.
- **Oblate:** An oblate spheroid has a surface area of $A_{\mathrm{S}} = 2\pi R_{\mathrm{R}}^2 + \frac{\pi\xi_{\mathrm{obl}}^2 R_{\mathrm{R}}^2}{\sqrt{1-\xi_{\mathrm{obl}}^2}} \ln\frac{1+\sqrt{1-\xi_{\mathrm{obl}}^2}}{1-\sqrt{1-\xi_{\mathrm{obl}}^2}}$. The cross section is the same as for the cylinder.
- **Torus:** The torus has $A_{\mathrm{S}} = 2\pi(R_{\mathrm{R}} - R_{\mathrm{H}})2\pi R_{\mathrm{H}} = 4\pi^2\xi_{\mathrm{torus}}R_{\mathrm{R}}^2(1 - \xi_{\mathrm{torus}})$ and $A_{\perp,\mathrm{H}} = \pi R_{\mathrm{R}}^2\left(1 - (1 - 2\xi_{\mathrm{torus}})^2\right)$.
- **Dumbbell:** The surface area of the dumbbell spheres is $A_{\mathrm{S}} = 4\pi R_{\mathrm{H}}^2 + 4\pi R'^2_{\mathrm{H}} = 4\pi(1+\nu^2)\xi_{\mathrm{db}}^2 R_{\mathrm{R}}^2$, projected onto $A_{\perp,\mathrm{H}} = \pi(1+\nu^2)\xi_{\mathrm{db}}^2 R_{\mathrm{R}}^2$
- **Dumbbell with Tube:** The combined dumbbell shape has a surface of $A_{\mathrm{S}} = 4\pi R_{\mathrm{H}}^2 + 4\pi R'^2_{\mathrm{H}} + 2\pi R_{\mathrm{T}} L_{\mathrm{T}} - 2\pi R_{\mathrm{T}}^2 = 2\pi R_{\mathrm{R}}^2\left(2(1+\nu^2)\xi_{\mathrm{db}}^2 + \xi_{\mathrm{T}} l_{\mathrm{T}} - \xi_{\mathrm{T}}^2\right)$. Its cross section is $\pi R_{\mathrm{R}}^2 \xi_{\mathrm{db}}^2(1+\nu^2) + 2R_{\mathrm{T}}L_{\mathrm{T}} = R_{\mathrm{R}}^2\left(\pi\xi_{\mathrm{db}}^2(1+\nu^2) + 2\xi_{\mathrm{T}} l_{\mathrm{T}}\right)$.

Since the cooling and lighting computations [10] assume a cylinder shape, an effective habitat radius as approximate input for them is derived from the cross section ($R_{\mathrm{H,eff}} = \sqrt{\frac{A_{\perp,\mathrm{H}}}{\pi}}$).

B Artificial Gravity

B.1 Rotation Rate

The maximum radius in the habitat R_{R} is the place of maximum gravity $g_{\max} = \omega^2 R_{\mathrm{R}}$. The rotation rate follows to be $\omega = \sqrt{\frac{g_{\max}}{R_{\mathrm{R}}}}$. A maximum rotation rate implies a minimum rotational radius (for given maximum gravity), and hence a minimum volume for a specific shape.

B.2 Interior Gravity Distribution

As gravity is $g = \omega^2 r = \frac{g_{\max}}{R_{\mathrm{R}}} r$ with radius r, finding the gravity distribution of volume and ground area is equivalent to finding its distribution over radius. The volumetric gravity distribution (fraction of volume per gravity bin) is $\frac{\Delta V}{V \Delta g}$. For small heights Δr, a volume element ΔV is proportional to the surface area $A(r)$ of a cylinder shell that intersects with the habitat, $\Delta V = A(r)\Delta r = A(r)\frac{R_{\mathrm{R}}}{g_{\max}}\Delta g$.

The total volume inside r is $\int_0^r A dr$. $A(r)$ depends on the habitat shape:

- **Cylinder:** For a cylinder, the circumference grows linearly with radius, $A(r) = 2\pi r L_{\mathrm{cyl}}$ with cylinder length L_{cyl}. Thus, the cylinder has a linear gravity distribution with the peak at maximum gravity (triangle shape), and three quarters of the volume above half the maximum gravity.

- **Tube:** For the central region of a tube ($r < R_{\mathrm{T}}$), the length of the cylinder shell varies between $2R_{\mathrm{T}}$ (in the plane along the tube) and $2\sqrt{R_{\mathrm{T}}^2 - r^2}$ (in the plane perpendicular to it). With the circumference $2\pi r$ and an average length, the area is $A(r) = 2\pi r\left(R_{\mathrm{T}} + \sqrt{R_{\mathrm{T}}^2 - r^2}\right)$. In the outer region ($r > R_{\mathrm{T}}$) the intersection of tube hull and cylinder shell is an ellipse with half widths of R_{T} and $r \arcsin \frac{R_{\mathrm{T}}}{r}$, so $A(r) = 2\pi r R_{\mathrm{T}} \arcsin \frac{R_{\mathrm{T}}}{r}$.

- **Oblate:** A sphere has $A(r) = 4\pi r\sqrt{R_{\mathrm{R}}^2 - r^2}$, as the length of the shells decreases with r. The peak is at $\frac{R_{\mathrm{R}}}{\sqrt{2}}$ (which follows from setting its derivative to 0). Integration of $A(r)$ yields $-\frac{4\pi}{3}\left(R_{\mathrm{R}}^2 - r^2\right)^{\frac{3}{2}}$, so almost two thirds of the volume are at above half maximum gravity ($\frac{3}{4}^{\frac{3}{2}}$). Oblates have the same distribution as the sphere, so $A(r) = 4\pi r \xi_{\mathrm{obl}}\sqrt{R_{\mathrm{R}}^2 - r^2}$.

- **Torus:** The torus has $A(r) = 4\pi r\sqrt{R_{\mathrm{H}}^2 - (r - R_{\mathrm{R}} + R_{\mathrm{H}})^2}$.

- **Dumbbell (smaller sphere):** A dumbbell allows the most flexibility by varying the connecting cable length and the volume ratio of the two spheres. If the dumbbell is asymmetric, the following holds only for the outer sphere, whose center is at $R_{\mathrm{R}} - R_{\mathrm{H}}$ from the axis (the analysis is equivalent for the larger sphere). The ground is an elliptic intersection of a cylindrical shell. The half-width parallel to the axis is $a_{\parallel} = \sqrt{R_{\mathrm{H}}^2 - (r - R_{\mathrm{R}} + R_{\mathrm{H}})^2}$. The other half-width is a section of a circle, with the same angle γ as in the triangle formed by the center of mass, the intersection of hull and circle, and the center of the sphere. As all lengths of the triangle are known (r, R_{H}, and $R_{\mathrm{R}} - R_{\mathrm{H}}$), this angle can be found by the cosine rule, $\cos\gamma = \frac{r^2 + (R_{\mathrm{R}} - R_{\mathrm{H}})^2 - R_{\mathrm{H}}^2}{2r(R_{\mathrm{R}} - R_{\mathrm{H}})} = \frac{r^2 + R_{\mathrm{R}}^2 - 2R_{\mathrm{R}}R_{\mathrm{H}}}{2r(R_{\mathrm{R}} - R_{\mathrm{H}})}$, so $a_{\perp} = r \arccos \frac{r^2 + R_{\mathrm{R}}^2 - 2R_{\mathrm{R}}R_{\mathrm{H}}}{2r(R_{\mathrm{R}} - R_{\mathrm{H}})}$. The ground area for the outer sphere is thus $A(r) = \pi a_{\parallel} a_{\perp} = \pi r\sqrt{R_{\mathrm{H}}^2 - (r - R_{\mathrm{R}} + R_{\mathrm{H}})^2} \arccos \frac{r^2 + R_{\mathrm{R}}^2 - 2R_{\mathrm{R}}R_{\mathrm{H}}}{2r(R_{\mathrm{R}} - R_{\mathrm{H}})}$.

- **Dumbbell (larger sphere):** In the above derivation, R_{R} is replaced by R'_{R} and R_{H} by R'_{H}. But if the center of mass is within the larger sphere (i.e. $R'_{\mathrm{R}} < 2R'_{\mathrm{H}}$) and $r < 2R'_{\mathrm{H}} - R'_{\mathrm{R}}$, the cylinder shell has a length between $a_{\parallel}$ and $a_{\parallel,\min} =$

$\sqrt{R_{\rm H}\prime^2 - (r + R'_{\rm R} - R'_{\rm H})^2}$, while $a_\perp = \pi r$. The flattened shape of the ground consists of two half-ellipses and a rectangle, giving $A(r) = \pi(a_\parallel - a_{\parallel,\rm min})a_\perp + 4a_{\parallel,\rm min}a_\perp$.

- **Dumbbell with tube:** If a tube connects the two spheres, the resulting solution is the sum of the dumbbell spheres and the tube. As the tube can be asymmetric, the maximum radius is $R_{\rm R} - 2R_{\rm H}$ on the small-sphere side and $R'_{\rm R} - 2R'_{\rm H}$ on the other side. If the center of mass lies within the larger sphere, the minimum radius is $2R'_{\rm H} - R'_{\rm R}$.

B.3 Hull Gravity Distribution

The gravity distribution at the hull is needed for computing the structural mass for holding the shielding.

The gravity distribution boils down to the hull surface distribution over radius r from the axis, $\Delta A_{\rm S}(r) = L(r) f_{\rm O}(r)\Delta r$ with the length of the intersection of surface and cylinder shell $L(r)$ and an orientation factor $f_{\rm O}(r)$ that takes the orientation of the surface relative to the cylinder shell into account. This factor is $\frac{1}{\cos\alpha}$ where α is the angle between surface and radius vector $\vec{r}$, so it is 1 where the hull surface is perpendicular to the axis ($\alpha = 0$), e.g. cylinder endcaps. Where the surface is parallel to the axis, the corresponding surface area is at only this radius, e.g. the cylinder rim at $r = R_{\rm R}$. Otherwise, the total surface inside r is $\int_0^r L f_{\rm O} \mathrm{d}r$.

- **Cylinder:** The cylinder has a surface of $2\pi R_{\rm R} L_{\rm cyl}$ at $r = R_{\rm R}$. The two endcaps have length $L(r) = 4\pi r$ and orientation factor $f_{\rm O} = 1$.

- **Tube:** Using the same approximations as for the tube interior area, and approximating the circumference of an ellipse as 2π times the square root of the average half-width squares, the hull length is (for two sides) $L(r) = 4\pi\sqrt{\frac{r^2+\left(R_{\rm T}\arcsin\frac{r}{R_{\rm T}}\right)^2}{2}}$ for $r < R_{\rm T}$ and $L(r) = 4\pi\sqrt{\frac{R_{\rm T}^2+\left(r\arcsin\frac{R_{\rm T}}{r}\right)^2}{2}}$ for $r > R_{\rm T}$. The orientation factor is approximated as 1 (it would be larger at $r \approx R_{\rm T}$)

- **Oblate:** A sphere has $L(r) = 4\pi r$ and $f_{\rm O} = \frac{R_{\rm R}}{\sqrt{R_{\rm R}^2 - r^2}}$. The oblate has the same L, and its surface orientation is found by $\Delta A_{\rm S} = L\sqrt{\Delta a^2 + \Delta r^2} = L\sqrt{1 + \left(\frac{\Delta a}{\Delta r}\right)^2}\Delta r$. Deriving the defining equation $r = \sqrt{R_{\rm R}^2 - \left(\frac{a}{\xi_{\rm obl}}\right)^2}$ with respect to a yields $f_{\rm O} = \sqrt{1 + \left(\frac{\Delta a}{\Delta r}\right)^2} = \sqrt{1 + \frac{r^2\xi_{\rm obl}^2}{R_{\rm R}^2 - r^2}}$.

- **Torus:** The length is $L(r) = 4\pi r$ (where $r > R_{\rm R} - 2R_{\rm H}$) and the orientation factor is $f_{\rm O} = \frac{R_{\rm H}}{\sqrt{R_{\rm H}^2 - (r - R_{\rm R} + R_{\rm H})^2}}$.

- **Dumbbell (smaller sphere):** The length is approximately $L = 2\pi\sqrt{\frac{a_\parallel^2 + a_\perp^2}{2}}$. The orientation factor parallel to the axis is $f_{\rm O,\parallel} = \frac{R_{\rm H}}{a_\parallel}$. Perpendicular to the axis, the angle β between $\vec{r}$ and $\vec{R_H}$ is found by the cosine rule, $\cos\beta = \frac{r^2+R_H^2-(R_R-R_H)^2}{2rR_H}$. As the hull surface is perpendicular to $\vec{R_H}$, $\alpha = \beta - \frac{\pi}{2}$ and $\cos\alpha = \sin\beta = \sqrt{1-\cos^2\beta}$. For combination with the parallel factor, $\frac{1}{\cos\alpha}$ is expressed as $\frac{R_H}{a_{\parallel,\text{eff}}}$ to yield an effective $a_{\parallel,\text{eff}} = R_H \sin\beta$. This enables an averaging which avoids too high factors if one a is very low: $f_O = \frac{2R_H}{a_\parallel + a_{\parallel,\text{eff}}}$.

- **Dumbbell (larger sphere):** Again, R_R is replaced by R'_R and R_H by R'_H. If $r < 2R'_H - R'_R$, the cylinder shell intersects the sphere along a length of approximately $L = 4\pi r \frac{\sqrt{(a_\parallel - a_{\parallel,\text{min}})^2+4r^2}}{2r}$. The $4\pi r$ is the circumference of the cylinder shell (on both sides), the additional factor takes the stretching along the axis into account. The orientation factor is approximated as $f_O = \frac{2R_H}{a_\parallel + a_{\parallel,\text{min}}}$.

B.4 Jumping Height

This section is only a sidenote and not directly relevant to the model, but could help in choosing floor heights. Shielding requirements demand low volume and thus low heights. On the other hand, enough space for the life forms must be provided. For humans, a ceiling at jumping height is usually sufficient; any visual impression can be displayed by screens and mirrors. Exceptions are places where flight devices are used (preferably in low-gravity sections), where balls are thrown, or where large trees grow.

The acceleration phase is important for a calculation of jumping heights at different gravities. A common misconception is to assume constant take-off velocity, which results in a height inversely proportional to gravity $h = \frac{g_E}{g} h_E$ (with the Earth values denoted by subscript $_E$). However, the muscle force must always provide the gravitational acceleration g in addition to dynamic acceleration a.

If instead assuming constant muscle force per body mass of $f = a + g$ over an acceleration length of s, the take-off velocity is $v = \sqrt{2sa} = \sqrt{2s(f-g)}$. The resulting jumping height is $h = \frac{v^2}{2g} = s\frac{f-g}{g}$. This is equivalent to a constant amount of work if s remains the same, $W = mgs + \frac{1}{2}mv^2 = mg(s+h) = mg_E(s+h_E)$, so relative to the Earth values the height is $h = \frac{g_E}{g}(s+h_E) - s$.

As the muscle force decreases with faster contraction, a different assumption could be constant (average) power $P = \frac{W}{t}$. In this case the work is reduced by the shorter time of acceleration, which is approximately inversely proportional to the take-off velocity. Therefore the quantity $Wv = mg(s+h)\sqrt{2gh} = m\sqrt{2g^3h}(s+h)$ is held constant, $\sqrt{g_E^3 h_E}(s+h_E) = \sqrt{g^3 h}(s+h)$. To determine h, a first approximation (neglecting s in the low-gravity case) is $h_0 = \frac{1}{g}\left(\sqrt{g_E^3 h_E}(s+h_E)\right)^{\frac{2}{3}} = \frac{g_E}{g} h_E^{\frac{1}{3}}(s+h_E)^{\frac{2}{3}}$. Using $s\sqrt{h} \approx s\sqrt{h_0}$ yields $h = \frac{1}{g}\left(\sqrt{g_E^3 h_E}(s+h_E) - s\sqrt{g^3 h_0}\right)^{\frac{2}{3}}$.

A detailed analysis of the leg muscle force at different velocities could give an accurate answer; presumably the constant power assumption comes closest. The velocity of maximum power is reached earlier at low gravity, but the final power could be lower.

For example, if on Earth $h_E = s = 0.5$m, the jumping height on the Moon ($\frac{g_E}{g} = 6$) is predicted to be

- for constant take-off velocity $h = 6h_\mathrm{E} = 3\mathrm{m}$.
- for constant force/work $h = 6(s + h_\mathrm{E}) - s = 5.5\mathrm{m}$.
- for constant power $h_0 = 6h_\mathrm{E}^{\frac{1}{3}}(s+h_\mathrm{E})^{\frac{2}{3}} = 4.8\mathrm{m}$ and $h = 0.6\left(10\sqrt{5} - 0.5 * 10/6\sqrt{48/6}\right)^{\frac{2}{3}} = 0.6 * 20^{\frac{2}{3}} = 4.4\mathrm{m}$.

The take-off velocity is $v = \sqrt{2gh}$, which amounts to 3.2, 4.3, and $3.8\frac{\mathrm{m}}{\mathrm{s}}$, respectively.

A jump (or throw) in a rotating habitat is a straight line as seen from a non-rotating rest frame. Maximum height is obtained by jumping backwards with respect to rotation, so that the direction of the jump is perpendicular to the resulting total velocity vector v_t. The angle between v_t and the rotational velocity v_r is determined by $\sin\alpha = \frac{v}{v_\mathrm{r}}$. The same angle is found between the radius vectors at start of the jump R (perpendicular to v_r) and at maximum height R_min (perpendicular to v_t), $\cos\alpha = \frac{R_\mathrm{min}}{R}$. The height is thus

$$h = R - R_\mathrm{min} = R\,(1 - \cos\alpha) = R\left(1 - \sqrt{1 - \frac{v^2}{Rg}}\right)$$

making use of $\cos^2 + \sin^2 = 1$ and $g = \frac{v_\mathrm{r}^2}{R}$ to replace v_r. For small v, the classical expression $h = \frac{v^2}{2g}$ is recovered by approximating $\sqrt{1 - \frac{v^2}{Rg}} \approx 1 - \frac{v^2}{2Rg}$. A jump through the center of rotation is possible if $v > v_\mathrm{r}$.

A vertical jump (jump direction perpendicular to v_r) is calculated using $\tan\alpha = \frac{v}{v_\mathrm{r}}$ to yield $h = R\left(1 - \cos\operatorname{atan}\frac{v}{\sqrt{Rg}}\right)$.

For example, with a take-off velocity of $3.2\frac{\mathrm{m}}{\mathrm{s}}$ in a habitat of 100m radius and $10\frac{\mathrm{m}}{\mathrm{s}^2}$ maximum gravity, one could jump up to $h = 3.42\mathrm{m}$ at lunar gravity ($R = 17\mathrm{m}$) instead of the classical 3m, but only to a height of 2.4m in a vertical jump. A rotating observer would explain the trajectory (roulette, [1]) with the Coriolis force.

B.5 Floor Height

The habitat is divided into floors or alternating layers of ground and air. A simple model for a possible variation of air height with gravity is described here.

Part of the air height (h_var) is inversely proportional to gravity ($h_\mathrm{var,min}$ at maximum gravity). This ensures a constant landing speed of something falling from that height. The other part (h_const) is independent of gravity and includes the ground height. The floor height is then $h(R) = h_\mathrm{const} + h_\mathrm{var,min}\frac{R_\mathrm{R}}{R}$, from which the radii of the floors are deduced starting at R_R.

Although jumping take-off speed is a bit higher at lower gravities, it is not necessary to increase the whole air height with lower gravity. Increasing only half of it ($h_\mathrm{var,min} = h_\mathrm{const}$) would triple the air height at lunar gravity. Even if the air height at maximum gravity is only 2.5m, the 7.5m would be enough for the relaxed motion constraints.

The gravity distribution of ground area (fraction of area per gravity bin) $\frac{\Delta A}{A\Delta g}$ is obtained by weighting the volumetric distribution by the density of floors. It can be skewed towards higher gravity if floor heights vary. For example, if the whole floor height is inversely proportional to radius, the density of floors increases linearly with radius, and a flat volumetric gravity distribution becomes a triangle shape with respect to ground surface. A cylinder has a quadratic distribution in this case.

The average gravity can be volume-weighted or ground-weighted. The former is derived by integrating $\int_0^{R_\mathrm{R}} A(r) r \mathrm{d}r$ and dividing the result by $V_\mathrm{H} = \int_0^{R_\mathrm{R}} A(r) \mathrm{d}r$. For the latter, all floor areas times r are added and divided by the total ground area.

C Structural Integrity

Structural material is required to stabilize the whole system against the effects of both air pressure and centrifugal force. It is characterized by its density ρ_W (subscript W for weight support) and by how much force per cross section is applied to it, which must be less than the tensile strength and is denoted σ_W. The results depend only on that ratio, the tensile stress per density $\Sigma = \frac{\sigma_\mathrm{W}}{\rho_\mathrm{W}}$ (which has units of $\frac{\mathrm{Nm}^{-2}}{\mathrm{kgm}^{-3}} = \frac{\mathrm{Nm}}{\mathrm{kg}} = \frac{\mathrm{J}}{\mathrm{kg}} = \frac{\mathrm{m}^2}{\mathrm{s}^2}$).

C.1 Air Pressure

The hull of a sphere of radius R and air pressure p_A has to withstand a stretching force of $\pi R^2 p_\mathrm{A}$ over its cross section $2\pi R d$ (with its thickness d), as the air of a half-sphere effectively presses outwards. This requires $\sigma_\mathrm{W} = \frac{\pi R^2 p_\mathrm{A}}{2\pi R d} = \frac{R p_\mathrm{A}}{2d}$. With $d = \frac{R p_\mathrm{A}}{2\sigma_\mathrm{W}}$, it results in a hull mass of $M_\mathrm{p,A} = \rho_\mathrm{W} 4\pi R^2 d = \frac{\rho_\mathrm{W} p_\mathrm{A}}{\sigma_\mathrm{W}} 2\pi R^3 = \frac{3 p_\mathrm{A}}{2\Sigma} V$ with $V = \frac{4\pi}{3} R^3$.

Other shapes need a bit more structural mass per volume. For example, a long tube of quadratic cross section with diameter D and length L requires at least $\sigma_\mathrm{W} = \frac{DLp_\mathrm{A}}{2Ld} = \frac{Dp_\mathrm{A}}{2d}$ and a mass of $M_\mathrm{p,A} = \rho_\mathrm{W} 4DLd = \frac{2p_\mathrm{A}}{\Sigma} V$ with $V = LD^2$.

Due to safety reasons, a habitat would not be a single air body, but consist of several segments that can contain the pressure independently. This resembles the latter case (quadratic cross section). The structural mass is proportional to the volume, adding a structural density to hold the air pressure $\rho_\mathrm{p,A} \approx \frac{2p_\mathrm{A}}{\Sigma}$.

The air mass of a completely air-filled habitat is $M_\mathrm{A} = V_\mathrm{H} \frac{p_\mathrm{A}}{10^5 \mathrm{Pa}} 1.2 \frac{\mathrm{kg}}{\mathrm{m}^3}$ with the air pressure p_A. This is included in the interior mass (which has to be at least this air mass).

C.2 Gravity Support

Artificial gravity (with acceleration $g = \omega^2 R$) can be countered by either vertical or (nearly) horizontal cables. The former connect a mass element to the rotation axis (hanging), the latter connect the masses at a certain radius with each other (standing). N mass points of mass ΔM at the same radius R require each either a cable length of R vertically holding $\Delta M \omega^2 R$, or a length of $2R \sin\frac{\pi}{N}$ horizontally holding $\frac{\Delta M \omega^2 R}{2\sin\frac{\pi}{N}}$, as can be seen geometrically (the force is half the projected force as there are two cables holding that mass point). For large N, the length is $2\pi R$ and the force $\frac{M\omega^2 R}{2\pi}$.

As the cross section is the force divided by σ_W ($\frac{\Delta M \omega^2 R}{\sigma_\mathrm{W}}$ vertically), the required structural mass is the product of density, cross section, and length:

$$\Delta M_\mathrm{W} = \frac{\rho_\mathrm{W} \Delta M \omega^2 R^2}{\sigma_\mathrm{W}} = \frac{g_\mathrm{max} R^2}{\Sigma R_\mathrm{R}} \Delta M$$

with $\omega^2 = \frac{g_{\max}}{R_{\mathrm{R}}}$. It is identical in both cases if self-weight is neglected. At $R = R_{\mathrm{R}}$, it is proportional to the radius and the inverse tensile stress.

C.3 Bridges between Vertical Cables

If pure horizontal support is possible, a circle around the axis can hold each floor. Otherwise (dumbbell and tube), vertical cables hold a floor structure that spans the gap between them. The need for free space rules out additional cables that connect the floor to the vertical cables (akin to a cable-stayed bridge). Instead, the support must be integrated in the horizontal floor. To estimate the structural mass, a simple suspension bridge is modeled, or equivalently a rigid arc structure. A mixture of these two ensures that no net horizontal forces act on the vertical cables as the suspension part pulls and the arc part pushes.

The vertical force diminishes with distance from the vertical cable r as $F_{\mathrm{V}}(r) = M_{\mathrm{out}}(r)g = \chi(r)Mg$ with the gravitational acceleration g. M_{out} is the supported mass outside of r. It equals the total supported mass M at the vertical cable ($F_{\mathrm{V}}(0) = Mg$) and falls off with the distribution function $\chi(r)$.

In equilibrium, the horizontal force is the same everywhere and given by $F_{\mathrm{H}} = F_{\mathrm{V}}(0)\tan\alpha = Mg\tan\alpha$, where α is the angle between vertical cable and bridge. The shape of the cable follows from $\frac{\Delta y}{\Delta r} = \frac{F_{\mathrm{V}}}{F_{\mathrm{H}}} = \frac{\chi(r)}{\tan\alpha}$. Integrating yields $y(r) = \frac{1}{\tan\alpha}\int_0^r \chi(r)\mathrm{d}r$.

The integration for y results in a parabolic shape if the supported mass decreases linearly with distance from the vertical cable ($\chi = 1-\frac{r}{R}$). But a uniform mass distribution over the supported area is assumed here, which means a quadratic decrease with radius, or $\chi = 1 - \left(\frac{r}{R}\right)^2$. The shape is thus $y(r) = \frac{1}{\tan\alpha}\left(r - \frac{r^3}{3R^2}\right)$.

With given distance between the vertical cables d_{V} (so $R \approx \frac{d_{\mathrm{V}}}{2}$) and bridge thickness (height of floor structure) b, the angle between vertical cable and bridge is derived from $b = \frac{1}{\tan\alpha}\left(\frac{d_{\mathrm{V}}}{2} - \frac{d_{\mathrm{V}}}{6}\right)$, or $\tan\alpha = \frac{d_{\mathrm{V}}}{3b}$.

The length differential is $\Delta L(r) = \sqrt{\Delta r^2 + \Delta y^2} = \Delta r\sqrt{1 + \frac{\chi(r)}{\tan\alpha}}$. The cross section is adapted to withstand the force, $A(r) = \frac{1}{\sigma_{\mathrm{W}}}\sqrt{F_{\mathrm{V}}^2 + F_{\mathrm{H}}^2} = \frac{Mg}{\sigma_{\mathrm{W}}}\sqrt{\chi^2 + \tan^2\alpha}$. The structural mass of the bridges is thus $M_{\mathrm{B}} = \rho_{\mathrm{W}}\int_0^R A(r)\frac{\Delta L}{\Delta r}\mathrm{d}r = \frac{\rho_{\mathrm{W}}Mg}{\sigma_{\mathrm{W}}}\int_0^R \sqrt{\chi^2 + \tan^2\alpha}\sqrt{1 + \frac{\chi(r)}{\tan\alpha}}\mathrm{d}r \approx \frac{\rho_{\mathrm{W}}Mg}{\sigma_{\mathrm{W}}}\int_0^R \left(\tan\alpha + \frac{\chi}{2}\right)\mathrm{d}r$. The approximation is valid if $\tan\alpha \gg \chi$, i.e. if the horizontal force dominates over the vertical force (much longer than thick). In this case, both expressions are roughly $\sqrt{\tan^2\alpha + \chi\tan\alpha}$. It follows that $M_{\mathrm{B}} = \frac{\rho_{\mathrm{W}}Mg}{\sigma_{\mathrm{W}}}\int_0^R \left(\tan\alpha + \frac{\chi}{2}\right)\mathrm{d}r = \frac{\rho_{\mathrm{W}}Mg}{\sigma_{\mathrm{W}}}\int_0^R \left(\frac{d_{\mathrm{V}}}{3b} + \frac{1-\left(\frac{r}{R}\right)^2}{2}\right)\mathrm{d}r = \frac{\rho_{\mathrm{W}}Mg}{\sigma_{\mathrm{W}}}\left(\frac{d_{\mathrm{V}}^2}{6b} + \frac{d_{\mathrm{V}}}{4} - \frac{d_{\mathrm{V}}}{12}\right) = \frac{\rho_{\mathrm{W}}Mgd_{\mathrm{V}}}{6\sigma_{\mathrm{W}}}\left(\frac{d_{\mathrm{V}}}{b} + 1\right)$.

C.4 Horizontal Self-Weight

The mass of the structural material itself must also be supported. The required cross section A for horizontal support can be derived analogously to air pressure (Sec. C.1). In a cylindrical rim of length L and radius R, the outward force $(M + M_{\mathrm{W}})\omega^2 R$ acts on the surface area of $2\pi RL$, creating a pressure of $p_{\mathrm{W}} = \frac{(M+M_{\mathrm{W}})\omega^2}{2\pi L}$. On the other hand, the stretching force on two opposite sides of the circular cable is the pressure times the area between the sides, $p_\mathrm{W}2RL$, acting on the cross sectional area $2A$, hence a tensile strength of $\sigma_\mathrm{W} = \frac{p_\mathrm{W}2RL}{2A}$ is needed. This yields a pressure of $p_\mathrm{W} = \frac{A\sigma_\mathrm{W}}{RL} = \frac{M_\mathrm{W}\Sigma}{2\pi R^2 L}$ (finding A from $M_\mathrm{W} = 2\pi RA\rho_\mathrm{W}$). Equating the two expressions for the pressure, $M\omega^2 = M_\mathrm{W}(\frac{\Sigma}{R^2} - \omega^2)$, gives

$$M_\mathrm{W} = M(\frac{\Sigma}{\omega^2 R^2} - 1)^{-1}$$

The same can be obtained from the earlier result that the force per mass along a horizontal cable is 2π times smaller than along a vertical one, which implies a cross section of $A = \frac{(M+M_\mathrm{W})\omega^2 R}{2\pi\sigma_\mathrm{W}}$ (again using $M_\mathrm{W} = 2\pi RA\rho_\mathrm{W}$).

If $\frac{\Sigma}{\omega^2 R^2} \gg 1$, the structural mass can be approximated as $M_\mathrm{W} = M\frac{\omega^2 R^2}{\Sigma}$, which is the result without self-weight.

C.5 Vertical Self-Weight

The gravity that acts on a vertical cable linearly decreases upwards. While the lower end holds only the mass element, the upper end additionally holds the cable and must therefore be thicker.

The force at radius R is $F(R) = M\omega^2 R$. It increases upwards by $\Delta F = -\omega^2 r\rho_\mathrm{W} A(r)\Delta r$, so the cross section increases by $\Delta A = \frac{\Delta F}{\sigma_\mathrm{W}} = -\frac{\omega^2}{\Sigma} rA(r)\Delta r$.

An ansatz is $A(r) = B\mathrm{e}^{-Cr^2}$, as its derivation resembles this expression: $\frac{\mathrm{d}A}{\mathrm{d}r} = -2rCA$. It follows that $C = \frac{\omega^2}{2\Sigma}$. Also, as $A(R) = \frac{F(R)}{\sigma_\mathrm{W}} = \frac{M\omega^2 R}{\sigma_\mathrm{W}} = B\mathrm{e}^{-CR^2}$, the remaining unknown is $B = \frac{M\omega^2 R}{\sigma_\mathrm{W}}\mathrm{e}^{CR^2}$, so

$$A(r) = \frac{M\omega^2 R}{\sigma_\mathrm{W}}\mathrm{e}^{-\frac{\omega^2}{2\Sigma}(R^2 - r^2)}$$

This is a Gaussian shape, centered at the axis with its maximum B and falling outwards until reaching $\frac{M\omega^2 R}{\sigma_\mathrm{W}}$ at R.

The structural mass is $M_\mathrm{W} = \rho_\mathrm{W}\int_0^R A(r)\mathrm{d}r = \rho_\mathrm{W} B\int_0^R \mathrm{e}^{-Cr^2}\mathrm{d}r = \rho_\mathrm{W} B\sqrt{\frac{\pi}{4C}}\operatorname{erf}\left(\sqrt{C}R\right) = \frac{\rho_\mathrm{W} B}{2C}\sqrt{\pi C}\operatorname{erf}\left(\sqrt{C}R\right)$, leading to the final result

$$M_\mathrm{W} = MR\sqrt{\pi C}\mathrm{e}^{CR^2}\operatorname{erf}\left(\sqrt{C}R\right)$$

The Gaussian error function $\operatorname{erf}\left(\sqrt{C}R\right)$ is approximately 1 for $\sqrt{C}R > 1$ and $2\sqrt{\frac{C}{\pi}}R$ for small $\sqrt{C}R$ ($\lesssim 0.5$). The former case describes an exponential explosion of structural mass with growing R. In the latter case, the exponential term approximates 1 and the expression reduces to $M_\mathrm{W} \approx 2MR^2C = M\frac{\omega^2 R^2}{\Sigma}$, which is the structural mass without self-weight.

The derivation is similar to the constant-stress solution for a space elevator, which also holds its own weight in varying gravity.

In addition to the payload mass, the bridges between the vertical cables have to be hold. The total structural mass is thus $M_\mathrm{W,total} = M_\mathrm{B} + \frac{M_\mathrm{W}}{M}(M + M_\mathrm{B})$. If horizontal support is possible due to a closed circle around the axis (in energy collection as well as in cylinder, oblate spheroid, and torus habitats), the lower mass of horizontal and vertical is chosen. Habitat interior and hull in dumbbell and tube need the vertical support with bridges. The radiator requires vertical support, but no bridges (using its hull for support).

C.6 Co-rotational Limit

Contrary to the space elevator, self-weight would not pose a problem to space habitats, as long as the radius of co-rotating components is not too large. A critical co-rotational radius R_{CoRot} is derived from the limit for horizontal support, $\frac{\Sigma}{\omega^2 R_{\mathrm{CoRot}}^2} = 1$, or

$$R_{\mathrm{CoRot}} = \sqrt{\frac{\Sigma}{\omega^2}} = \sqrt{\frac{1}{2C}} = \sqrt{\frac{\Sigma R_{\mathrm{R}}}{g_{\mathrm{max}}}}$$

In terms of $r = \frac{R}{R_{\mathrm{CoRot}}}$, the structural mass can be expressed as $M_{\mathrm{W}} = M\left(r^{-2} - 1\right)^{-1}$ for horizontal support and $M_{\mathrm{W}} = Mr\sqrt{\frac{\pi}{2}}\mathrm{e}^{\frac{r^2}{2}}\,\mathrm{erf}\left(\sqrt{\frac{1}{2}}r\right)$ for vertical support. Without self-weight, it is $M_{\mathrm{W}} = Mr^2$. The bridge mass is $M_{\mathrm{B}} = Mr\frac{d_{\mathrm{V}}}{6R_{\mathrm{CoRot}}}\left(\frac{d_{\mathrm{V}}}{b} + 1\right)$.

At the co-rotational limit ($r = 1$), horizontal support is not possible, and vertical support results in $M_{\mathrm{W}} = M\sqrt{\frac{\pi e}{2}}\,\mathrm{erf}\left(\sqrt{\frac{1}{2}}\right) = 1.4M$, while $M_{\mathrm{W}} = M$ without self-weight. At twice that radius ($r = 2$), the mass could still be hold vertically, but at the expense of $M_{\mathrm{W}} = M2\sqrt{\frac{\pi}{2}}e^2\,\mathrm{erf}\left(2\sqrt{\frac{1}{2}}\right) = 17.6M$. At half that radius ($r = \frac{1}{2}$), $M_{\mathrm{W}} = \frac{M}{3}$ for horizontal and $M_{\mathrm{W}} = MR\sqrt{\pi C}\mathrm{e}^{CR^2}\,\mathrm{erf}\left(\sqrt{C}R\right) = M\sqrt{\frac{\pi}{8}}\mathrm{e}^{\frac{1}{8}}\,\mathrm{erf}\left(\sqrt{\frac{1}{8}}\right) = \frac{M}{4}\mathrm{e}^{\frac{1}{8}} = 0.28M$ for vertical support. Without self-weight, it would be $\frac{M}{4}$.

$M_{\mathrm{W}} = M$ is at $r = 0.88$ for vertical support (at 0.71 for horizontal support and at 1 without self-weight). At $r = 0.31$, structural mass amounts to 10%, and at $r = 0.1$ to only 1% of M.

The rotational radius with a certain r is found by setting $R = R_{\mathrm{R}}$, so $R_{\mathrm{R}} = rR_{\mathrm{CoRot}} = r^2\frac{\Sigma}{g_{\mathrm{max}}}$. This makes 10km for $r = 1$, $\Sigma = 10^5\,\frac{\mathrm{Nm}}{\mathrm{kg}}$ and Earth gravity. For $r = 0.1$ $R_{\mathrm{R}} = 100$m and $R_{\mathrm{CoRot}} = 1$km.

Co-rotation can only be maintained if the radius is not too large, thus a maximum radius for light and electricity is given by $\gamma_{\mathrm{L,E}}R_{\mathrm{CoRot}}$. Above that radius, the energy collection system is decoupled from rotation. Mirrors are preferably kept in rotation to allow higher concentration, while non-rotating PV modules can be connected by a sliding contact at the axis. Although decoupling from rotation could be achieved for heat emission as well, sliding contacts at the central connection tube are more difficult due to the pressure. The radiator is hence assumed to co-rotate. To maintain structural integrity, its maximum radius R_{em} is the minimum of the quadratic solution, $\kappa_{\mathrm{em}}R_{\mathrm{R}}$ (as in [10]), and $\gamma_{\mathrm{em}}R_{\mathrm{CoRot}}$, where γ_{em} is the maximum ratio of radiator to co-rotational radius.

C.7 Computation

The total structural mass is obtained by integrating over the radius. In the accompanying program, this is done numerically by summing the contributions from the floors (using the ground radius for the interior mass and an average radius for the hull).

Radiator, mirrors, and electricity generation are separated into N slices of equal mass. As the expression without self-weight is proportional to the square of the radius, an effective radius of such a slice is derived by analytical integration, $R_{\mathrm{eff,i}}^2 = \int_{R_\mathrm{i}}^{R_{\mathrm{i+1}}} r^2 \frac{\mathrm{d}M}{M_\mathrm{i}}$.

The radiator has a rectangular shape, so the incremental mass increase is independent of r, $\frac{\mathrm{d}M}{M_\mathrm{i}} = \frac{\mathrm{d}r}{R_{\mathrm{i+1}} - R_\mathrm{i}}$. With $R_\mathrm{i} = i\frac{R_\mathrm{em}}{N}$, integration results in

$$R_{\mathrm{eff,i}} = \sqrt{\frac{R_{\mathrm{i+1}}^3 - R_\mathrm{i}^3}{3(R_{\mathrm{i+1}} - R_\mathrm{i})}} = \sqrt{\frac{(i+1)^3 - i^3}{3}}\frac{R_\mathrm{em}}{N}$$

For the circular energy collection system, $\frac{\mathrm{d}M}{M_\mathrm{i}} = \frac{2r\mathrm{d}r}{R_{\mathrm{i+1}}^2 - R_\mathrm{i}^2}$, and the integration leads to $R_{\mathrm{eff,i}}^2 = \frac{2}{R_{\mathrm{i+1}}^2 - R_\mathrm{i}^2}\int_{R_\mathrm{i}}^{R_{\mathrm{i+1}}} r^3 \mathrm{d}r = \frac{R_{\mathrm{i+1}}^4 - R_\mathrm{i}^4}{2(R_{\mathrm{i+1}}^2 - R_\mathrm{i}^2)} = \frac{R_{\mathrm{i+1}}^2 + R_\mathrm{i}^2}{2}$. Equal areas of circular slices are obtained by $R_\mathrm{i} = \sqrt{R_\mathrm{min}^2 + i\frac{R_\mathrm{max}^2 - R_\mathrm{min}^2}{N}}$, so

$$R_{\mathrm{eff,i}} = \sqrt{\frac{R_{\mathrm{i+1}}^2 + R_\mathrm{i}^2}{2}} = \sqrt{R_\mathrm{min}^2 + (i + \frac{1}{2})\frac{R_\mathrm{max}^2 - R_\mathrm{min}^2}{N}}$$

R_min is 0 for light and R_L for electricity, and R_max is R_L for light and R_E for electricity (or $\gamma_{\mathrm{L,E}} R_\mathrm{CoRot}$, if that is smaller).

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